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THE STROH FORMALISM FOR ANISOTROPIC MATERIALS THAT POSSESS AN ALMOST EXTRAORDINARY DEGENERATE MATRIX N

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Abstract—It has been discovered recently that the 6×6 elasticity matrix **N** for an anisotropic elastic material under a two-dimensional deformation can be *extraordinary degenerate*, i.e., **N** can have three identical pairs of complex conjugate eigenvalues but has only one pair of complex conjugate eigenvector. The Stroh formalism and the modified formalism for a simple degenerate **N** available in the literature need to be revised. What is more likely to happen in applications is that the matrix **N** is almost extraordinary degenerate. In that case the orthonormalized eigenvectors can be very large, causing certain numerical problems. We therefore present in this paper a modified formalism that is applicable to the case when **N** can be extraordinary degenerate or almost extraordinary degenerate. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The sextic formalism of Stroh for anisotropic elasticity is based on the assumption that the 6×6 real matrix **N** is simple or semisimple so that the three pairs of complex conjugate eigenvectors ξ_α span a six-dimensional space. When **N** is non-semisimple or degenerate, i.e., when the number of independent eigenvectors is less than three pairs, the Stroh formalism does not apply. Ting and Hwu (1988) and Ting (1992) have introduced modified formalisms that apply to a degenerate or almost degenerate **N** that has two pair of complex conjugate eigenvectors. Isotropic material is an example. When there exists only one pair of complex conjugate eigenvector, **N** is extraordinary degenerate. Since isotropic material is the most degenerate material, in the physical sense, of all anisotropic materials, most researchers believe that an extraordinary degenerate **N** does not exist. Recently Ting (1996) has proved that an extraordinary degenerate **N** exists. Hence isotropic material is not the most degenerate material in the mathematical sense. For an extraordinary degenerate material, the Stroh formalism and the modified formalism proposed by Ting and Hwu are not valid. The difficulty occurs not only when **N** is extraordinary degenerate. When **N** is almost degenerate it is shown in Ting and Hwu (1988) that the magnitude of an orthonormalized eigenvector can be very large, and becomes infinite as **N** becomes degenerate. This may cause problems in a numerical computation. We therefore present in this paper a modified formalism that is valid regardless of whether **N** is extraordinary degenerate or almost extraordinary degenerate.

2. THE SEXTIC FORMALISM OF STROH

In a Cartesian coordinate system x_i, the equations of equilibrium in terms of the displacements u_i are

$$C_{ijkl}u_{k,lj} = 0 \tag{1}$$

in which repeated indices imply summation, a comma stands for differentiation and C_{ijkl} are the elasticity constants with the symmetry property

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$$C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij}. \tag{2}$$

For two-dimensional deformations in which u_k ($k = 1, 2, 3$) depends on x_1 and x_2 only, a general solution of (1) is

$$u_k = a_k f(z), \quad z = x_1 + p x_2, \tag{3}$$

where $f(z)$ is an arbitrary function of z, and a_k and p are determined by inserting (3) into (1). In matrix notation we have

$$\mathbf{\Gamma}(p)\mathbf{a} = 0 \tag{4}$$

in which

$$\mathbf{\Gamma}(p) = \mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}, \tag{5}$$

the superscript T denotes the transpose, and the 3×3 matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are given by

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{6}$$

It can be shown that the six eigenvalues p_α ($\alpha = 1, 2, \ldots, 6$) of (4) cannot be real if the strain energy is positive (Eshelby *et al.*, 1953). Thus p_α consists of three pairs of complex conjugates, as do their associated eigenvectors $\mathbf{a}_\alpha$. Without loss in generality we let

$$\operatorname{Im}\{p_\alpha\} > 0, \quad p_{\alpha+3} = \bar{p}_\alpha, \quad \mathbf{a}_{\alpha+3} = \bar{\mathbf{a}}_\alpha, \quad (\alpha = 1, 2, 3) \tag{7}$$

where Im stands for the imaginary part and the overbar denotes the complex conjugate. The general solution for the displacement vector $\mathbf{u}$ obtained by superposing six solutions of the form (3) can be written as

$$\mathbf{u} = \sum_{\alpha=1}^{3} [\mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha-3}(z_\alpha)] \tag{8}$$

in which $f_\alpha(z_\alpha)$ are arbitrary functions of their argument and

$$z_\alpha = x_1 + p_\alpha x_2. \tag{9}$$

Since $\mathbf{u}$ must be real, we let

$$f_{\alpha+3}(z_\alpha) = \overline{f_\alpha(z_\alpha)} = \bar{f}_\alpha(\bar{z}_\alpha). \tag{10}$$

Introducing the vector $\mathbf{b}$ by

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a} \tag{11}$$

where the second equality comes from (4), and letting

$$\phi = \sum_{\alpha=1}^{3} [\mathbf{b}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{b}}_\alpha \overline{f_\alpha(z_\alpha)}], \tag{12}$$

the stresses σ_{ij} obtained from

$$\sigma_{ij} = C_{ijkl} u_{k,l} \tag{13}$$

can be written as

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}. \tag{14}$$

Equations $(11)_1$ and $(11)_2$ can be rewritten in the standard eigenrelation as

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{15}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{16}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^{\mathrm{T}}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} = \mathbf{N}_3^{\mathrm{T}}. \tag{17}$$

Thus $\boldsymbol{\xi}$ is a right eigenvector of the 6×6 real matrix $\mathbf{N}$. The left eigenvector $\boldsymbol{\eta}$ satisfies

$$\mathbf{N}^{\mathrm{T}}\boldsymbol{\eta} = p\boldsymbol{\eta}. \tag{18}$$

Introducing the 6×6 matrix $\mathbf{J}$ by

$$\mathbf{J} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix} \tag{19}$$

where $\mathbf{I}$ is the 3×3 identity matrix, it can be shown that

$$\mathbf{J}\mathbf{N} = (\mathbf{J}\mathbf{N})^{\mathrm{T}} = \mathbf{N}^{\mathrm{T}}\mathbf{J}. \tag{20}$$

From (15), (18) and (20) we may assume without loss in generality (Chadwick and Smith 1977)

$$\boldsymbol{\eta} = \mathbf{J}\boldsymbol{\xi} = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}. \tag{21}$$

When $\mathbf{N}$ is simple or semisimple, ξ_α spans a six-dimensional space and is orthogonal to $\boldsymbol{\eta}_\alpha$. Thus we may normalize $\boldsymbol{\xi}_\alpha$ such that (with $\boldsymbol{\eta}_\alpha$ determined from (21))

$$\boldsymbol{\eta}_\beta^{\mathrm{T}}\boldsymbol{\xi}_\alpha = \delta_{\alpha\beta} \tag{22}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The orthonormal relations can be written in matrix notation as

$$\mathbf{V}^{\mathrm{T}}\mathbf{U} = \mathbf{I} \tag{23}$$

in which the 6×6 matrices $\mathbf{U}$ and $\mathbf{V}$ are

$$\left.\begin{aligned} \mathbf{U} &= [\boldsymbol{\xi}_1, \quad \boldsymbol{\xi}_2, \quad \boldsymbol{\xi}_3, \quad \bar{\boldsymbol{\xi}}_1, \quad \bar{\boldsymbol{\xi}}_2, \quad \bar{\boldsymbol{\xi}}_3], \\ \mathbf{V} &= [\boldsymbol{\eta}_1, \quad \boldsymbol{\eta}_2, \quad \boldsymbol{\eta}_3, \quad \bar{\boldsymbol{\eta}}_1, \quad \bar{\boldsymbol{\eta}}_2, \quad \bar{\boldsymbol{\eta}}_3]. \end{aligned}\right\} \tag{24}$$

If we introduce the 3×3 matrices

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{25}$$

we may write **U** and **V** as

$$\mathbf{U} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix}, \quad \mathbf{V} = \mathbf{J}\mathbf{U} = \begin{bmatrix} \mathbf{B} & \bar{\mathbf{B}} \\ \mathbf{A} & \bar{\mathbf{A}} \end{bmatrix}. \tag{26}$$

Equation (23) implies that $\mathbf{V}^{\mathrm{T}}$ and $\mathbf{U}$ are the inverses of each other and their product can be interchanged. Carrying out the matrix multiplication of the interchanged product leads to

$$\left.\begin{aligned} \mathbf{A}\mathbf{A}^{\mathrm{T}} + \bar{\mathbf{A}}\bar{\mathbf{A}}^{\mathrm{T}} &= 0 = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \bar{\mathbf{B}}\bar{\mathbf{B}}^{\mathrm{T}}, \\ \mathbf{B}\mathbf{A}^{\mathrm{T}} + \bar{\mathbf{B}}\bar{\mathbf{A}}^{\mathrm{T}} &= \mathbf{I} = \mathbf{A}\mathbf{B}^{\mathrm{T}} + \bar{\mathbf{A}}\bar{\mathbf{B}}^{\mathrm{T}}. \end{aligned}\right\} \tag{27}$$

These are the closure relations. Equation (27) tell us that the three Barnett–Lothe tensors **S**, **H**, **L**, defined by

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^{\mathrm{T}}, \tag{28}$$

are real. Clearly **H** and **L** are symmetric, and can be shown to be positive definite if the strain energy is positive (Chadwick and Smith, 1977). The three Barnett–Lothe tensors play important roles in the problems of anisotropic elasticity and surface waves (see, e.g., Chadwick and Smith, 1977; Barnett and Lothe, 1973, 1974, 1975; Ting, 1986). They satisfy the identity

$$\mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{I}. \tag{29}$$

The above formalism is valid if **N** is simple or semisimple. When **N** is almost degenerate, say p_1 and p_2 are almost equal as are ξ_1 and ξ_2, Ting and Hwu (1988) have shown that the orthonormalized eigenvectors ξ_1 and ξ_2 are very large and become infinite when **N** is degenerate. To overcome this difficulty they proposed a modified formalism that applies to **N** that is degenerate or almost degenerate.

When **N** is extraordinary degenerate, i.e., when $p_1 = p_2 = p_3$ and $\xi_1 = \xi_2 = \xi_3$, neither the Stroh formalism nor the modified formalism proposed by Ting and Hwu is valid. It has been widely conjectured that an extraordinary degenerate material does not exist. Recently Ting (1996) has shown that the conjecture is incorrect. In fact, the set of extraordinary degenerate **N** is probably larger than the set of degenerate **N**, of which isotropic material is a special case. In the following sections we present a modified formalism that applies to extraordinary degenerate or almost extraordinary degenerate materials.

3. MODIFIED SEXTIC FORMALISM

In this section, we assume that p_1, p_2 and p_3 are either equal or almost equal, so are the corresponding eigenvectors. From (15) let

$$\left.\begin{aligned} \mathbf{N}\xi_1^0 &= p_1\xi_1^0, \\ \mathbf{N}\xi_2^0 &= p_2\xi_2^0, \\ \mathbf{N}\xi_3^0 &= p_3\xi_3^0, \end{aligned}\right\} \tag{30}$$

in which ξ_α^0 need not be normalized. After subtracting $(30)_1$ from $(30)_2$ and dividing the resulting equation by $(p_2 - p_1)$ we have

$$\mathbf{N}\left(\frac{\xi_2^0-\xi_1^0}{p_2-p_1}\right)=p_2\left(\frac{\xi_2^0-\xi_1^0}{p_2-p_1}\right)+\xi_1^0. \tag{31}$$

Similarly, from $(30)_1$ and $(30)_3$ we obtain

$$\mathbf{N}\left(\frac{\xi_3^0-\xi_1^0}{p_3-p_1}\right)=p_3\left(\frac{\xi_3^0-\xi_1^0}{p_3-p_1}\right)+\xi_1^0. \tag{32}$$

Again, subtracting (31) from (32) we have

$$\mathbf{N}\left[\left(\frac{\xi_3^0-\xi_1^0}{p_3-p_1}\right)-\left(\frac{\xi_2^0-\xi_1^0}{p_2-p_1}\right)\right]=p_3\left[\left(\frac{\xi_3^0-\xi_1^0}{p_3-p_1}\right)-\left(\frac{\xi_2^0-\xi_1^0}{p_2-p_1}\right)\right]+(p_3-p_2)\left(\frac{\xi_2^0-\xi_1^0}{p_2-p_1}\right). \tag{33}$$

Equations $(30)_1$, (31) and (33) give us a new system of equations

$$\left.\begin{aligned}\mathbf{N}\xi_1' &= p_1\xi_1',\\ \mathbf{N}\xi_2' &= p_2\xi_2'+\xi_1',\\ \mathbf{N}\xi_3' &= p_3\xi_3'+\xi_2',\end{aligned}\right\} \tag{34}$$

where

$$\left.\begin{aligned}&\xi_1'=\xi_1^0,\quad \xi_2'=\frac{(\xi_2^0-\xi_1^0)}{\delta_3},\\ &\xi_3'=\frac{1}{\delta_1}\left(\frac{\xi_3^0-\xi_1^0}{-\delta_2}-\frac{\xi_2^0-\xi_1^0}{\delta_3}\right)=\frac{\delta_1\xi_1^0+\delta_2\xi_2^0+\delta_3\xi_3^0}{-\delta_1\delta_2\delta_3},\end{aligned}\right\} \tag{35}$$

$$\delta_1=p_3-p_2,\quad \delta_2=p_1-p_3,\quad \delta_3=p_2-p_1. \tag{36}$$

In $(35)_3$ we have used the relation

$$\delta_1+\delta_2+\delta_3=0. \tag{37}$$

For the left eigenvectors

$$\mathbf{N}^{\mathrm{T}}\boldsymbol{\eta}_i^0=p_i\boldsymbol{\eta}_i^0,\quad (i=1,2,3) \tag{38}$$

similar analysis but in the reverse order for $i=3,2,1$ yields

$$\left.\begin{aligned}\mathbf{N}^{\mathrm{T}}\boldsymbol{\eta}_3' &= p_3\boldsymbol{\eta}_3',\\ \mathbf{N}^{\mathrm{T}}\boldsymbol{\eta}_2' &= p_2\boldsymbol{\eta}_2'+\boldsymbol{\eta}_3',\\ \mathbf{N}^{\mathrm{T}}\boldsymbol{\eta}_1' &= p_1\boldsymbol{\eta}_1'+\boldsymbol{\eta}_2',\end{aligned}\right\} \tag{39}$$

where

$$\boldsymbol{\eta}_3'=\boldsymbol{\eta}_3^0,\quad \boldsymbol{\eta}_2'=\frac{\boldsymbol{\eta}_3^0-\boldsymbol{\eta}_2^0}{\delta_1},\quad \boldsymbol{\eta}_1'=\frac{\delta_1\boldsymbol{\eta}_1^0+\delta_2\boldsymbol{\eta}_2^0+\delta_3\boldsymbol{\eta}_3^0}{-\delta_1\delta_2\delta_3}. \tag{40}$$

Equations (35), (37) and (40) also give us

$$\xi_1^0 = \xi_1', \quad \xi_2^0 = \xi_1' + \delta_3 \xi_2', \quad \xi_3^0 = \xi_1' - \delta_2 \xi_2' - \delta_1 \delta_2 \xi_3', \tag{41}$$

$$\eta_3^0 = \eta_3', \quad \eta_2^0 = \eta_3' - \delta_1 \eta_2', \quad \eta_1^0 = \eta_3' + \delta_2 \eta_2' - \delta_2 \delta_3 \eta_1'. \tag{42}$$

Thus, in the modified formalism, we will use ξ_α' and η_α' instead of ξ_α^0 and η_α^0. They are determined from (34) and (39). The vectors ξ_α^0 and η_α^0 are not employed, but their relations with ξ_α' and η_α' as given by (41) and (42) will be useful in establishing certain identities. Instead of solving (39) for η_α', they can be obtained from ξ_α'. From (21) we have

$$\eta_k^0 = \mathbf{J}\xi_k^0, \quad (k = 1, 2, 3) \tag{43}$$

With the use of (40), (43) and (41) it can be shown that

$$\eta_1' = \mathbf{J}\xi_3', \quad \eta_2' = \mathbf{J}\xi_2' - \delta_2 \mathbf{J}\xi_3', \quad \eta_3' = \mathbf{J}\xi_1' - \delta_2 \mathbf{J}\xi_2' - \delta_1 \delta_2 \mathbf{J}\xi_3', \tag{44}$$

or, in matrix form,

$$[\eta_1', \eta_2', \eta_3'] = \mathbf{J}[\xi_1', \xi_2', \xi_3']\mathbf{Y} \tag{45}$$

in which

$$\mathbf{Y} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\delta_2 \\ 1 & -\delta_2 & -\delta_1 \delta_2 \end{bmatrix} = \mathbf{Y}^{\mathrm{T}}. \tag{46}$$

The eigenvectors ξ_α' and η_α' obtained from (34) and (39) are not unique. We will show that one can obtain a set of vectors which are orthonormal. Let $\tilde{\xi}_\alpha'$ and $\tilde{\eta}_\alpha'$ be a solution and let

$$\xi_1' = k_1 \tilde{\xi}_1', \quad \xi_2' = k_2 \tilde{\xi}_2' + k_2' \tilde{\xi}_1', \quad \xi_3' = k_3 \tilde{\xi}_3' + k_3' \tilde{\xi}_2' + k_3'' \tilde{\xi}_1', \tag{47}$$

where k_1, k_2, k_3, k_2', k_3', and k_3'' are complex constants to be determined. In matrix form we have

$$[\xi_1', \xi_2', \xi_3'] = [\tilde{\xi}_1', \tilde{\xi}_2', \tilde{\xi}_3']\mathbf{K} \tag{48}$$

in which

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2' & k_3'' \\ 0 & k_2 & k_3' \\ 0 & 0 & k_3 \end{bmatrix}. \tag{49}$$

Insertion of (47) into (34) leads to the following relations

$$k_2 = k_1 + \delta_3 k_2', \quad k_3 = k_2 + \delta_1 k_3' = k_1 - \delta_2 k_2' - \delta_1 \delta_2 k_3'', \quad k_3' = k_2' - \delta_2 k_3''. \tag{50}$$

From (45) and (48) we have

$$[\eta_1', \eta_2', \eta_3'] = \mathbf{J}[\tilde{\xi}_1', \tilde{\xi}_2', \tilde{\xi}_3']\mathbf{K}\mathbf{Y} = [\tilde{\eta}_1', \tilde{\eta}_2', \tilde{\eta}_3']\mathbf{Y}^{-1}\mathbf{K}\mathbf{Y}. \tag{51}$$

With $\mathbf{Y}$ and $\mathbf{K}$ given by (46) and (49), it can be shown that

$$\mathbf{KY} = \begin{bmatrix} k_3'' & k_3' & k_3 \\ k_3' & k_2 - \delta_2 k_3' & -\delta_2 k_3 \\ k_3 & -\delta_2 k_3 & -\delta_1 \delta_2 k_3 \end{bmatrix} \tag{52}$$

which is symmetric. Hence

$$\mathbf{KY} = \mathbf{YK}^{\mathrm{T}}, \quad \text{or} \quad \mathbf{Y}^{-1}\mathbf{KY} = \mathbf{K}^{\mathrm{T}}. \tag{53}$$

Thus (51) is simplified to

$$[\boldsymbol{\eta}_1', \boldsymbol{\eta}_2', \boldsymbol{\eta}_3'] = [\tilde{\boldsymbol{\eta}}_1', \tilde{\boldsymbol{\eta}}_2', \tilde{\boldsymbol{\eta}}_3']\mathbf{K}^{\mathrm{T}}. \tag{54}$$

To have an orthonormal system we need

$$[\boldsymbol{\eta}_1', \boldsymbol{\eta}_2', \boldsymbol{\eta}_3']^{\mathrm{T}}[\boldsymbol{\xi}_1', \boldsymbol{\xi}_2', \boldsymbol{\xi}_3'] = \mathbf{I} \tag{55}$$

or, by (48) and (54),

$$\mathbf{K}[\tilde{\boldsymbol{\eta}}_1', \tilde{\boldsymbol{\eta}}_2', \tilde{\boldsymbol{\eta}}_3']^{\mathrm{T}}[\tilde{\boldsymbol{\xi}}_1', \tilde{\boldsymbol{\xi}}_2', \tilde{\boldsymbol{\xi}}_3']\mathbf{K} = \mathbf{I}. \tag{56}$$

This means that

$$[\mathbf{K}^{-1}]^2 = [\tilde{\boldsymbol{\eta}}_1', \tilde{\boldsymbol{\eta}}_2', \tilde{\boldsymbol{\eta}}_3']^{\mathrm{T}}[\tilde{\boldsymbol{\xi}}_1', \tilde{\boldsymbol{\xi}}_2', \tilde{\boldsymbol{\xi}}_3']. \tag{57}$$

From (49) we have

$$\mathbf{K}^{-1} = \begin{bmatrix} \dfrac{1}{k_1} & -\dfrac{k_2'}{k_1 k_2} & \dfrac{k_2' k_3' - k_2 k_3''}{k_1 k_2 k_3} \\ 0 & \dfrac{1}{k_2} & -\dfrac{k_3'}{k_2 k_3} \\ 0 & 0 & \dfrac{1}{k_3} \end{bmatrix} \tag{58}$$

so that

$$[\mathbf{K}^{-1}]^2 = \begin{bmatrix} \dfrac{1}{k_1^2} & -\dfrac{k_2'(k_1+k_2)}{k_1^2 k_2^2} & \dfrac{k_2' k_3'(k_1 k_2 + k_2 k_3 + k_3 k_1) - k_2^2 k_3''(k_1+k_3)}{k_1^2 k_2^2 k_3^2} \\ 0 & \dfrac{1}{k_2^2} & -\dfrac{k_3'(k_2+k_3)}{k_2^2 k_3^2} \\ 0 & 0 & \dfrac{1}{k_3^2} \end{bmatrix}. \tag{59}$$

Equation (57) can be written explicitly as

$$\left.\begin{aligned} & k_1^{-2} = \tilde{\boldsymbol{\eta}}_1'^{\mathrm{T}}\tilde{\boldsymbol{\xi}}_1', \quad k_2^{\,2} = \tilde{\boldsymbol{\eta}}_2'^{\mathrm{T}}\tilde{\boldsymbol{\xi}}_2', \quad k_3^{-2} = \tilde{\boldsymbol{\eta}}_3'^{\mathrm{T}}\tilde{\boldsymbol{\xi}}_3', \\ & k_2' = -k_1^2 k_2^2 \tilde{\boldsymbol{\eta}}_1'^{\mathrm{T}}\tilde{\boldsymbol{\xi}}_2'/(k_1+k_2), \quad k_3' = -k_2^2 k_3^2 \tilde{\boldsymbol{\eta}}_2'^{\mathrm{T}}\tilde{\boldsymbol{\xi}}_3'/(k_2+k_3), \\ & k_3'' = [-k_1^2 k_2^2 k_3^2 \tilde{\boldsymbol{\eta}}_1'^{\mathrm{T}}\tilde{\boldsymbol{\xi}}_3' + k_2' k_3'(k_1 k_2 + k_2 k_3 + k_3 k_1)]/k_2^2(k_1+k_3). \end{aligned}\right\} \tag{60}$$

With the aid of (34) and (39) one can show that it is compatible with (50).

When $\delta_k \neq 0$, the orthogonality relations of ξ_α^0, η_α^0 and (35) and (40) assure us that k_α exist and do not vanish. In (60) one can always choose the signs of k_α so that $(k_\alpha + k_\beta) \neq 0$. Hence k_2', k_3' and k_3'' also exist.

When $\delta_k = 0$, (60) can be written with the aid of (50) as

$$\left.\begin{aligned} k_1^{-2} &= k_2^{-2} = k_3^{-2} = \bar{\eta}_1'^{\mathrm{T}}\bar{\xi}_1' = \bar{\eta}_2'^{\mathrm{T}}\bar{\xi}_2' = \bar{\eta}_3'^{\mathrm{T}}\bar{\xi}_3', \\ k_2' &= k_3' = -k_1^3\bar{\eta}_1'^{\mathrm{T}}\bar{\xi}_2'/2 = -k_1^3\bar{\eta}_2'^{\mathrm{T}}\bar{\xi}_3'/2, \\ k_3'' &= [-k_1^4\bar{\eta}_1'^{\mathrm{T}}\bar{\xi}_3' + 3k_2'^2]/2k_1. \end{aligned}\right\} \quad (61)$$

The existence of a set of orthonormalized eigenvectors is assured by the theory of non-semisimple matrices (Pease, 1965).

4. EIGENRELATIONS FOR $\mathbf{a}_\alpha'$ AND $\mathbf{b}_\alpha'$

The Stroh eigenrelation was in fact based on the earlier version, (4) and (11) proposed by Eshelby *et al.* (1953). Thus instead of finding the 6-vector ξ from (15) one could find the 3-vectors $\mathbf{a}$ and $\mathbf{b}$ from (4) and (11). This may have some advantages in a numerical calculation because each equation in (4) and (11) consists of three, not six, scalar equations. To modify (4) and (11), we follow the derivation of (34). We obtain

$$\left.\begin{aligned} \mathbf{\Gamma}(p_1)\mathbf{a}_1' &= 0, \\ \mathbf{\Gamma}(p_2)\mathbf{a}_2' &= -[(\mathbf{R}+\mathbf{R}^{\mathrm{T}})+(p_1+p_2)\mathbf{T}]\mathbf{a}_1', \\ \mathbf{\Gamma}(p_3)\mathbf{a}_3' &= -[(\mathbf{R}+\mathbf{R}^{\mathrm{T}})+(p_3+p_2)\mathbf{T}]\mathbf{a}_2' - \mathbf{T}\mathbf{a}_1'. \end{aligned}\right\} \quad (62)$$

As to the modification of (11), we have

$$\left.\begin{aligned} \mathbf{b}_1' &= (\mathbf{R}^{\mathrm{T}}+p_1\mathbf{T})\mathbf{a}_1' = -\left(\frac{1}{p_1}\mathbf{Q}+\mathbf{R}\right)\mathbf{a}_1', \\ \mathbf{b}_2' &= (\mathbf{R}^{\mathrm{T}}+p_2\mathbf{T})\mathbf{a}_2' + \mathbf{T}\mathbf{a}_1' = -\left(\frac{1}{p_2}\mathbf{Q}+\mathbf{R}\right)\mathbf{a}_2' + \frac{1}{p_1p_2}\mathbf{Q}\mathbf{a}_1', \\ \mathbf{b}_3' &= (\mathbf{R}^{\mathrm{T}}+p_3\mathbf{T})\mathbf{a}_3' + \mathbf{T}\mathbf{a}_2' = -\left(\frac{1}{p_3}\mathbf{Q}+\mathbf{R}\right)\mathbf{a}_3' + \frac{1}{p_2p_3}\mathbf{Q}\mathbf{a}_2' - \frac{1}{p_1p_2p_3}\mathbf{Q}\mathbf{a}_1'. \end{aligned}\right\} \quad (63)$$

Equations (62) and (63) provide $\mathbf{a}_\alpha'$ and $\mathbf{b}_\alpha'$ which form the components of ξ_α'. One then finds η_α' from (45) and orthonormalize the eigenvectors as outlined in Section 3. With (63), (62) can be rewritten as

$$\left.\begin{aligned} \mathbf{\Gamma}(p_1)\mathbf{a}_1' &= 0, \\ \mathbf{\Gamma}(p_2)\mathbf{a}_2' &= -(\mathbf{R}+p_2\mathbf{T})\mathbf{a}_1' - \mathbf{b}_1', \\ \mathbf{\Gamma}(p_3)\mathbf{a}_3' &= -(\mathbf{R}+p_3\mathbf{T})\mathbf{a}_2' - \mathbf{b}_2'. \end{aligned}\right\} \quad (64)$$

Therefore we may employ (64) and (63) instead of (62) and (63).

5. THE BARNETT–LOTHE TENSORS

With orthonormalized ξ_α' and η_α' we have

$$\mathbf{V}'^{\mathrm{T}}\mathbf{U}' = \mathbf{U}'\mathbf{V}'^{\mathrm{T}} = \mathbf{I} \quad (65)$$

in which

$$\left.\begin{aligned}\mathbf{U}' &= [\boldsymbol{\xi}'_1, \boldsymbol{\xi}'_2, \boldsymbol{\xi}'_3, \bar{\boldsymbol{\xi}}'_1, \bar{\boldsymbol{\xi}}'_2, \bar{\boldsymbol{\xi}}'_3],\\ \mathbf{V}' &= [\boldsymbol{\eta}'_1, \boldsymbol{\eta}'_2, \boldsymbol{\eta}'_3, \bar{\boldsymbol{\eta}}'_1, \bar{\boldsymbol{\eta}}'_2, \bar{\boldsymbol{\eta}}'_3].\end{aligned}\right\} \tag{66}$$

If we introduce the 3×3 matrices

$$\mathbf{A}' = [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3], \quad \mathbf{B}' = [\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3], \tag{67}$$

and use (45), we have

$$\mathbf{U}' = \begin{bmatrix} \mathbf{A}' & \bar{\mathbf{A}}' \\ \mathbf{B}' & \bar{\mathbf{B}}' \end{bmatrix}, \quad \mathbf{V}' = \begin{bmatrix} \mathbf{B}'\mathbf{Y} & \bar{\mathbf{B}}'\bar{\mathbf{Y}} \\ \mathbf{A}'\mathbf{Y} & \bar{\mathbf{A}}'\bar{\mathbf{Y}} \end{bmatrix}. \tag{68}$$

Carrying out the matrix multiplications in $(65)_2$ leads to

$$\left.\begin{aligned}\mathbf{A}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}} + \bar{\mathbf{A}}'\bar{\mathbf{Y}}\bar{\mathbf{A}}'^{\mathrm{T}} &= 0 = \mathbf{B}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} + \bar{\mathbf{B}}'\bar{\mathbf{Y}}\bar{\mathbf{B}}'^{\mathrm{T}},\\ \mathbf{B}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}} + \bar{\mathbf{B}}'\bar{\mathbf{Y}}\bar{\mathbf{A}}'^{\mathrm{T}} &= \mathbf{I} = \mathbf{A}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} + \bar{\mathbf{A}}'\bar{\mathbf{Y}}\bar{\mathbf{B}}'^{\mathrm{T}}.\end{aligned}\right\} \tag{69}$$

These are modified closure relations for (27). Since (27) leads to (28), one may assume that the three Barnett–Lothe tensors **S**, **H** and **L** have the form

$$\mathbf{H} = 2i\mathbf{A}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}}, \quad \mathbf{L} = -2i\mathbf{B}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}}, \quad \mathbf{S} = i(2\mathbf{A}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} - \mathbf{I}). \tag{70}$$

We will prove it in the following. It should be pointed out that in the special case $p_1 = p_2 = p_3$, $(70)_2$ reduces to the one established by Barnett (1992).

When **N** is simple, Lothe and Barnett (1976) have shown that

$$\langle \mathbf{N} \rangle \mathbf{U}' = i \begin{bmatrix} \mathbf{A}' & -\bar{\mathbf{A}}' \\ \mathbf{B}' & -\bar{\mathbf{B}}' \end{bmatrix} \tag{71}$$

in which

$$\langle \mathbf{N} \rangle = \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix}. \tag{72}$$

The identity remains valid for an extraordinary degenerate **N** (Wang, 1996). If we post-multiply both sides of (71) by $\mathbf{V}'^{\mathrm{T}}$ and use $(65)_2$ and (72), it leads to eqns (70).

If (70) hold for any δ_k, comparison with (28) suggests that the following conversion relations hold

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}}, \quad \mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{B}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}}, \quad \mathbf{A}\mathbf{B}^{\mathrm{T}} = \mathbf{A}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}}. \tag{73}$$

To prove it, we will derive the relations between $\boldsymbol{\xi}'_\alpha$ and $\boldsymbol{\xi}_\alpha$. Since the un-normalized eigenvectors $\boldsymbol{\xi}^0_\alpha$ are scalar multiples of $\boldsymbol{\xi}_\alpha$, we let

$$\boldsymbol{\xi}^0_1 = \alpha\boldsymbol{\xi}_1, \quad \boldsymbol{\xi}^0_2 = \beta\boldsymbol{\xi}_2, \quad \boldsymbol{\xi}^0_3 = \gamma\boldsymbol{\xi}_3, \tag{74}$$

in which α, β and γ are constants to be determined. Obviously, from (21), we also have

$$\boldsymbol{\eta}^0_1 = \alpha\boldsymbol{\eta}_1, \quad \boldsymbol{\eta}^0_2 = \beta\boldsymbol{\eta}_2, \quad \boldsymbol{\eta}^0_3 = \gamma\boldsymbol{\eta}_3. \tag{75}$$

From (35) and (40) we have

$$\left.\begin{aligned} \xi_1' &= \alpha\xi_1, \quad \xi_2' = \frac{\beta\xi_2-\alpha\xi_1}{\delta_3}, \quad \xi_3' = \frac{\delta_1\alpha\xi_1+\delta_2\beta\xi_2+\delta_3\gamma\xi_3}{-\delta_1\delta_2\delta_3}, \\ \boldsymbol{\eta}_3' &= \gamma\boldsymbol{\eta}_3, \quad \boldsymbol{\eta}_2' = \frac{\gamma\boldsymbol{\eta}_3-\beta\boldsymbol{\eta}_2}{\delta_1}, \quad \boldsymbol{\eta}_1' = \frac{\delta_1\alpha\boldsymbol{\eta}_1+\delta_2\beta\boldsymbol{\eta}_2+\delta_3\gamma\boldsymbol{\eta}_3}{-\delta_1\delta_2\delta_3}. \end{aligned}\right\} \tag{76}$$

Using the orthonormal relations of $\boldsymbol{\xi}_\alpha$, $\boldsymbol{\eta}_\alpha$ and $\boldsymbol{\xi}_\alpha'$, $\boldsymbol{\eta}_\alpha'$ it is easy to find that

$$\alpha^2 = -\delta_2\delta_3, \quad \beta^2 = -\delta_1\delta_3, \quad \gamma^2 = -\delta_1\delta_2. \tag{77}$$

From the identity in (37) we also have

$$\beta^2+\gamma^2 = \delta_1^2, \quad \alpha^2+\gamma^2 = \delta_2^2, \quad \alpha^2+\beta^2 = \delta_3^2. \tag{78}$$

Hence we have from (76)

$$\xi_1' = \alpha\xi_1, \quad \xi_2' = \delta_2\alpha^{-1}\xi_1-\delta_1\beta^{-1}\xi_2, \quad \xi_3' = \alpha^{-1}\xi_1+\beta^{-1}\xi_2+\gamma^{-1}\xi_3. \tag{79}$$

It tells us that

$$\mathbf{A}' = \mathbf{AE}, \quad \mathbf{B}' = \mathbf{BE}, \tag{80}$$

where

$$\mathbf{E} = \begin{bmatrix} \alpha & \delta_2\alpha^{-1} & \alpha^{-1} \\ 0 & -\delta_1\beta^{-1} & \beta^{-1} \\ 0 & 0 & \gamma^{-1} \end{bmatrix}. \tag{81}$$

By a direct calculation it can be shown that

$$\mathbf{EYE}^{\mathrm{T}} = \mathbf{I}. \tag{82}$$

Equations (80) and (82) lead to the identities in (73). Therefore, (70) hold for any δ_k. It is also useful to know that

$$\mathbf{E}^{-1} = \mathbf{YE}^{\mathrm{T}} = \begin{bmatrix} \alpha^{-1} & \beta^{-1} & \gamma^{-1} \\ 0 & \delta_3\beta^{-1} & -\delta_2\gamma^{-1} \\ 0 & 0 & \gamma \end{bmatrix}. \tag{83}$$

6. CONVERSION FROM THE STROH FORMALISM TO THE MODIFIED FORMALISM

With (73) and (80) one can convert relations which are valid for the Stroh formalism to relations for the modified formalism. For instance, the impedance tensor **M** is defined as (Ingebrigsten and Tonning, 1969; Chadwick and Ting, 1987)

$$\mathbf{M} = -i\mathbf{BA}^{-1} = \mathbf{H}^{-1}+i\mathbf{H}^{-1}\mathbf{S}. \tag{84}$$

or

$$\mathbf{M} = -i\mathbf{BA}^{-1} = -i\mathbf{B}'\mathbf{A}'^{-1} = -i\tilde{\mathbf{B}}'\tilde{\mathbf{A}}'^{-1} = \mathbf{H}^{-1}+i\mathbf{H}^{-1}\mathbf{S} \tag{85}$$

where, by (48),

$$\mathbf{A}' = \tilde{\mathbf{A}}'\mathbf{K}, \quad \mathbf{B}' = \tilde{\mathbf{B}}'\mathbf{K}. \tag{86}$$

Equation (85) is one of the few relations for which the conversion is achieved by a simple replacement of **A**, **B** by **A**′, **B**′ or $\tilde{\mathbf{A}}'$, $\tilde{\mathbf{B}}'$. Similarly, we also have

$$\mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = i\mathbf{A}'\mathbf{B}'^{-1} = i\tilde{\mathbf{A}}'\tilde{\mathbf{B}}'^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}. \tag{87}$$

Equations (85) and (87) tell us that we may use the un-orthonormalized $\tilde{\mathbf{A}}'$, $\tilde{\mathbf{B}}'$ in computing the impedence tensor **M** and its inverse $\mathbf{M}^{-1}$.

In most applications f_α in (8) and (12) assume the same function which can be written as

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha) \quad (\alpha \text{ not summed}) \tag{88}$$

where q_α are complex constants. Equations (8) and (12) can then be written in matrix form as

$$\mathbf{u} = 2\,\mathrm{Re}\,\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\}, \quad \boldsymbol{\phi} = 2\,\mathrm{Re}\,\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}, \tag{89}$$

in which

$$\mathbf{q}^{\mathrm{T}} = [q_1, q_2, q_3]$$

and

$$\langle f(z_*)\rangle = \mathrm{diag}\,[f(z_1), f(z_2), f(z_3)] \tag{90}$$

is a diagonal matrix. For the modified formalism, let

$$\mathbf{q}' = \mathbf{E}^{-1}\mathbf{q}. \tag{91}$$

We may write **u** and $\boldsymbol{\phi}$ as

$$\mathbf{u} = 2\,\mathrm{Re}\,\{\mathbf{A}'\mathbf{F}\mathbf{q}'\}, \quad \boldsymbol{\phi} = 2\,\mathrm{Re}\,\{\mathbf{B}'\mathbf{F}\mathbf{q}'\}, \tag{92}$$

where

$$\mathbf{F} = \mathbf{E}^{-1}\langle f(z_*)\rangle\mathbf{E}. \tag{93}$$

Carrying out the matrix product in (93) leads to

$$\mathbf{F} = \begin{bmatrix} f(z_1) & x_2 f'(z_2) & \frac{1}{2}x_2^2 f''(z_3) \\ 0 & f(z_2) & x_2 f'(z_3) \\ 0 & 0 & f(z_3) \end{bmatrix} \tag{94}$$

in which

$$\left.\begin{aligned} f'(z_2) &= \frac{f(z_2)-f(z_1)}{z_2-z_1}, \quad f'(z_3) = \frac{f(z_3)-f(z_2)}{z_3-z_2}, \\ f''(z_3) &= -2\frac{(z_3-z_2)f(z_1)+(z_2-z_1)f(z_3)+(z_1-z_3)f(z_2)}{(z_3-z_2)(z_2-z_1)(z_1-z_3)}. \end{aligned}\right\} \tag{95}$$

In the limit $p_3 = p_2 = p_1, f'(z)$ and $f''(z)$ in (95) are the first and second derivatives of $f(z)$, respectively. By carrying out the matrix product it can be shown that

$$\mathbf{KF} = \mathbf{FK}. \tag{96}$$

Thus, with the aid of (86), we may rewrite (92) as

$$\mathbf{u} = 2\,\mathrm{Re}\{\tilde{\mathbf{A}}'\mathbf{F}\tilde{\mathbf{q}}'\}, \quad \boldsymbol{\phi} = 2\,\mathrm{Re}\{\tilde{\mathbf{B}}'\mathbf{F}\tilde{\mathbf{q}}'\}, \tag{97}$$

where

$$\tilde{\mathbf{q}}' = \mathbf{K}\mathbf{q}'. \tag{98}$$

This means that $\mathbf{u}$, $\boldsymbol{\phi}$ is unique regardless of whether we use the orthonormalized $\mathbf{A}'$, $\mathbf{B}'$ or the un-orthonormalized $\tilde{\mathbf{A}}'$, $\tilde{\mathbf{B}}'$.

7. CONCLUDING REMARKS

The modified sextic formalism presented here applies to any matrix $\mathbf{N}$ which is almost extraordinary degenerate or extraordinary degenerate. The δ_k in the analysis need not be small. Hence it applies also to a simple $\mathbf{N}$. However, it cannot be applied to a semisimple or degenerate $\mathbf{N}$. Instead of the integral formalism (Barrett and Lothe, 1973), (70) offers an algebraic formalism for obtaining the Barnett–Lothe tensors $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ for an extraordinary degenerate $\mathbf{N}$. Another alternate is to employ (85) or (87) in conjunction with the identity (29). The alternate allows us to use the un-orthonormalized $\tilde{\mathbf{A}}'$ and $\tilde{\mathbf{B}}'$. With (92) or (97), if one knows the analytic solution for the material with a simple $\mathbf{N}$, one can easily convert it to the solution for the material with an extraordinary degenerate $\mathbf{N}$.

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